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The Clebsch–Gordan coefficients for the two-parameter quantum algebra $SU_{p,q}(2)$ in the Löwdin–Shapiro approach

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Abstract. The structure of the irreducible representations of the two-parameter quantum algebra $SU_{p,q}(2)$ is studied. The projection operators for this algebra are constructed in the Löwdin–Shapiro form. The explicit analytical expressions for the $SU_{p,q}(2)$ Clebsch–Gordan coefficients are obtained with the help of these projection operators. There are clear perspectives to elaborate the theory of tensor operators, universal R-matrix, $6j$, $9j$ symbols, etc for the $SU_{p,q}(2)$ algebra using the same tools.

1. Introduction

Quantum algebras were introduced in [1, 2] in connection with the inverse scattering problem. This concept was then developed in detail in [3–5] and by other authors [6–12]. The q-analogues of the Wigner–Racah algebra have been introduced and studied in [6–18]. In particular the angular momentum theory for the SU(2) and $SU_q(2)$ algebras was considered in [19–27]. In recent years special interest has been aroused about two-parameter quantum algebras [28–32]. The general features of the representation theory for the simplest two-parameter algebra $SU_{p,q}(2)$ were studied. However, the Clebsch–Gordan problem was not analysed. Here we apply the projection operator method, developed in [23, 24, 33] for standard Lie algebras and used in [25, 26] for the one-parameter quantum algebra $SU_q(2)$, to the solution of the Clebsch–Gordan problem. The main advantage of the projection operator method lies in the fact that for the calculation of quantities of the Wigner–Racah algebra no explicit realization of the generators of the quantum algebra is necessary. Only the commutation relations of the generators, their properties with respect to Hermitian conjugation and the existence of the highest weight vectors are sufficient for the development of the theory of unitary irreducible representation for the quantum algebra under consideration. The analysis given below shows as expected that the results for the $SU_{p,q}(2)$ irreducible representations are similar to the corresponding formulae for the one-parameter $SU_q(2)$ algebra [25–27] and hence to those of the standard angular momentum theory after the substitution of the numbers x by the p, q-numbers $[[x]]_{pq}$,

$$[[x]]_{pq} = [[x]] := \frac{q^x - p^{-x}}{q - p^{-1}} \tag{1.1}$$

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except for some factors containing powers of p and q, where p, q are assumed independent, real positive numbers. In the limit $p = q$ the $\mathrm{SU}_{p,q}(2)$ algebra transforms into the one-parameter $\mathrm{SU}_q(2)$ quantum algebra. The case of $p = q = 1$ corresponds to the standard SU(2) Lie algebra.

The paper is organized in the following manner. In section 2 the structure of the irreducible representation of the $\mathrm{SU}_{p,q}(2)$ algebra is discussed and the matrices of its generators are obtained in explicit form. In section 3 the projection operators for this quantum algebra are derived in a form of power series in the J_+, J_- generators, i.e. in the Löwdin–Shapiro form. The problem of 'vector coupling' of the $\mathrm{SU}_{p,q}(2)$ 'angular momenta' is discussed in section 4. The general analytical formula for the $\mathrm{SU}_{p,q}(2)$ Clebsch–Gordan coefficients is derived in section 5 by using the projection operator approach. The general scheme of calculations is similar to the corresponding procedure, developed for the one-parameter quantum algebra $\mathrm{SU}_q(2)$ in [25, 26]. The possibility to construct the theory of the $\mathrm{SU}_{p,q}(2)$ tensor operators, $6j, 9j \ldots$ symbols, universal R-matrix etc by this method becomes evident.

2. $\mathrm{SU}_{p,q}(2)$ algebra and its irreducible representations

The $\mathrm{SU}_{p,q}(2)$ algebra is defined by the three generators J_0, J_+, J_- with the following properties [28]:

$$
\begin{aligned}
&[J_0, J_\pm] = \pm J_\pm \\
&[J_+, J_-]_{pq} := J_+J_- - pq^{-1}J_-J_+ = [[2J_0]] \\
&J_0^\dagger = J_0 \qquad J_\pm^\dagger = J_\mp .
\end{aligned}
\tag{2.1}
$$

Here we use the notation

$$
[[2J_0]] = \frac{q^{2J_0} - p^{-2J_0}}{q - p^{-1}} .
$$

The finite-dimensional unitary irreducible representation (IR) D^j contains the highest weight vector $|jj\rangle$ satisfying the equations

$$
J_0 \mid jj\rangle = j \mid jj\rangle \qquad J_+ \mid jj\rangle = 0 \qquad \langle jj \mid jj\rangle = 1 . \tag{2.2}
$$

Using the generator J_- a non-normalized basis vector of the IR with weight m is constructed in the standard way,

$$
|jm) = J_-^n \mid jj\rangle \qquad \text{with} \quad m = j - n . \tag{2.3}
$$

The squared norm of this vector

$$
N^2(n) := (jm \mid jm) = \langle jj \mid J_+^n J_-^n \mid jj\rangle
$$

can be calculated from the next relation that can be proven by induction:

$$
J_+J_-^n = (pq^{-1})^n J_-^n J_+ + [[n]] J_-^{n-1}[[2J_0 - n + 1]](pq^{-1})^{n-1} . \tag{2.4}
$$

It follows that

$$N^2(n) = [[n]]\,[[2j-n+1]]\,N^2(n-1)(pq^{-1})^{n-1}\,. \tag{2.5}$$

The representation will be finite-dimensional only if a maximal value $n_{\max}$ exists, with $N^2(n_{\max}+1)=0$. This is the case when $2j - n_{\max} = 0$, i.e. $j = \frac{1}{2}n_{\max}$, hence only integer or half-integer values of the highest weight j, labelling the IRS D^j of the quantum algebra $SU_{p,q}(2)$ are admissible. In analogy with quantum mechanics we call j an 'angular momentum'. For its 'projection', m, the following values are possible: $m = j, j-1, \ldots, -j$. Thus the enumeration of the IRS of $SU_{p,q}(2)$, the weight structure and the dimensions, $\dim D^j = 2j+1$, of these IRS are the same as for the standard SU(2) algebra. Iterating (2.5) we obtain

$$N^2(n) = \frac{[[n]]!\,[[2j]]!}{[[2j-n]]!}(pq^{-1})^{\frac{1}{2}[n(n-1)]}\,. \tag{2.6}$$

It follows that the orthonormal basis vectors of D^j are given by

$$|jm\rangle = \sqrt{\frac{[[j+m]]!}{[[2j]]![[j-m]]!}}(pq^{-1})^{-\frac{1}{4}(j-m)(j-m-1)}\,J_-^{j-m}\mid jj\rangle\,. \tag{2.7}$$

By acting the generator J_- on the vector $|jm\rangle$ we obtain

$$J_-\mid j\,m\rangle = (pq^{-1})^{\frac{1}{2}(j-m)}\sqrt{[[j+m]][[j-m+1]]}\mid j\,m-1\rangle \tag{2.8}$$

and the application of J_+ to the vector $|jm\rangle$ gives

$$J_+\mid j\,m\rangle = (pq^{-1})^{\frac{1}{2}(j-m-1)}\sqrt{[[j-m]][[j+m+1]]}\mid j\,m+1\rangle\,. \tag{2.9}$$

Hence the explicit form of the IR of $SU_{p,q}(2)$,

$$\begin{aligned}
\langle jm'\mid J_0\mid jm\rangle &= m\,\delta_{m',m}\\
\langle jm'\mid J_-\mid jm\rangle &= (pq^{-1})^{\frac{1}{2}(j-m)}\sqrt{[[j+m]][[j-m+1]]}\,\delta_{m',m-1}\\
\langle jm'\mid J_+\mid jm\rangle &= (pq^{-1})^{\frac{1}{2}(j-m-1)}\sqrt{[[j-m]][[j+m+1]]}\,\delta_{m',m+1}
\end{aligned} \tag{2.10}$$

coincides with the corresponding formulae of the standard SU(2) algebra except for the substitution of the numbers $(j\pm m)$ and $(j\mp m+1)$ by the p,q-numbers $[[j\pm m]]$ and $[[j\mp m+1]]$ respectively. As for the powers of the generators we have

$$\begin{aligned}
\langle jm'\mid J_-^n\mid jm\rangle &= (pq^{-1})^{\frac{1}{4}[2n(j-m)+n(n-1)]}\sqrt{\frac{[[j+m]]!\,[[j-m+n]]!}{[[j-m]]!\,[[j+m-n]]!}}\,\delta_{m',m-n}\\
\langle jm'\mid J_+^n\mid jm\rangle &= (pq^{-1})^{\frac{1}{4}[2n(j-m)-n(n+1)]}\sqrt{\frac{[[j-m]]!\,[[j+m+n]]!}{[[j+m]]!\,[[j-m-n]]!}}\,\delta_{m',m+n}\,.
\end{aligned} \tag{2.11}$$

In the standard theory of angular momentum the vectors $|jm\rangle$ are the eigenvectors of the Casimir operator. This operator, characterized by its commutativity with all the generators of the algebra, takes for $\mathrm{SU}_{p,q}(2)$ the form

$$\begin{aligned} C_2 = J_- J_+ (pq^{-1})^{J_0} + (pq^{-1})^{J_0-1}[[J_0+\tfrac{1}{2}]]^2 \\ = J_+ J_- (pq^{-1})^{J_0-1} + (pq^{-1})^{J_0-2}[[J_0-\tfrac{1}{2}]]^2 . \end{aligned} \tag{2.12}$$

For its eigenvalues we have

$$C_2 \mid jm\rangle = \Lambda \mid jm\rangle \qquad \text{with} \quad \Lambda = (pq^{-1})^{j-1}[[j+\tfrac{1}{2}]]^2 . \tag{2.13}$$

To prove the relations appearing in this section the identities given in appendix A are useful.

3. Projection operators for the $\mathrm{SU}_{p,q}(2)$ algebra

In this section we introduce the main tool of the paper, the projection operators. Let us first consider the projection operator $P^j_{jj} \equiv P^j$ having the property

$$P^j \mid j'j\rangle = \delta_{j',j} \mid jj\rangle \tag{3.1}$$

i.e. the operator P^j projects into the subspace expanded by the highest weight vector of the IR D^j, $|jj\rangle$. As in [26] we seek this projector as a power series in the generators J_+ and J_-,

$$P^j = \sum_{l=0}^{\infty} c_l J_-^l J_+^l . \tag{3.2}$$

Note that the exponents of the generators J_-, J_+ must be the same due to the condition $[P^j, J_0] = 0$. Since $P^j \mid jj\rangle = |jj\rangle$ it follows that

$$c_0 = 1 \qquad \text{and} \qquad J_+ P^j = 0 . \tag{3.3}$$

Using (2.4) the following recurrent relation for the coefficients c_l is found:

$$c_{l-1} + [[l]]\,[[2j+l+1]]\,c_l = 0 . \tag{3.4}$$

It follows that

$$c_l = (-1)^l \frac{[[2j+1]]!}{[[l]]![[2j+l+1]]!} . \tag{3.5}$$

The problem of the convergence of the formal series (3.2) is not essential because the sum is always applied to vectors containing in their expansion only components $|j'j\rangle$ with $j' \leqslant j_{\max}$. Hence only a finite number of terms in the series gives a non-vanishing contribution, namely all terms with $l \leqslant j_{\max} - j$.

Using the properties of J_- and J_+ it is easy to check the Hermiticity of P^j. In a similar manner projecting P^j_{-j-j} into the subspace expanded by the lowest weight vector of the IR D^j, $|j-j\rangle$ can be constructed.

To deal with the Clebsch–Gordan problem a generalization of the projection operator is required. Let $P^j_{m,m'}$ be the operator defined by

$$P^j_{m,m'} \mid j'm''\rangle = \delta_{m',m''}\,\delta_{j,j'} \mid jm\rangle\,. \tag{3.6}$$

Thus this operator cancels all components except for $|jm'\rangle$ and then transforms it into the component $|jm\rangle$. Because of its projecting property we also refer to these operators as projection operators, although the exact term strictly applies only to the idempotent operators P^j_{mm}.

The formula defining these Hermitian operators is gained in a similar fashion as for P^j. Using (2.11) we obtain

$$\begin{aligned} P^j_{m,m'} = {} & (pq^{-1})^{\frac{1}{4}(j-m)(j-m-1)}\sqrt{\frac{[[j+m]]!}{[[2j]]![[j-m]]!}}\,J_-^{j-m}\,P^j\,J_+^{j-m'} \\ & \times\sqrt{\frac{[[j+m']]!}{[[2j]]![[j-m']]!}}(pq^{-1})^{\frac{1}{4}(j-m')(j-m'-1)}\,. \end{aligned} \tag{3.7}$$

From this equation it follows that

$$(P^j_{m,m'})^\dagger = P^j_{m',m}\,. \tag{3.8}$$

4. 'Vector coupling of angular momenta'

It is known that the expansion of the tensor product of the SU(2) IRS $D^{j_1}\otimes D^{j_2}$ into irreducible components is of the form

$$D^{j_1}\otimes D^{j_2} = \sum_{j=|j_1-j_2|}^{j_1+j_2} D^j \tag{4.1}$$

and that the generators acting in the tensor product space are given in terms of the generators in the initial spaces by

$$J_0(1,2) = J_0(1)+J_0(2) \qquad \text{and} \qquad J_\pm(1,2) = J_\pm(1)+J_\pm(2)\,. \tag{4.2}$$

Now for the $\mathrm{SU}_{p,q}(2)$ algebra the above relations become [28]

$$J_0(1,2) = J_0(1)+J_0(2) \qquad \text{and} \qquad J_\pm(1,2) = q^{J_0(1)}J_\pm(2)+J_\pm(1)p^{-J_0(2)} \tag{4.3}$$

or in the standard notation for Hopf algebras

$$\Delta(J_0) = J_0\otimes I + I\otimes J_0 \qquad \text{and} \qquad \Delta(J_\pm) = q^{J_0}\otimes J_\pm + J_\pm\otimes p^{-J_0}\,. \tag{4.4}$$

From these expressions the action of $\Delta(J_0)$ and $\Delta(J_\pm)$ on the vectors $|j_1m_1\rangle \otimes |j_2m_2\rangle \equiv |j_1m_1, j_2m_2\rangle$ can be seen to be given by

$$J_0(1,2) \mid j_1m_1, j_2m_2\rangle = (m_1+m_2) \mid j_1m_1, j_2m_2\rangle$$

and

$$\begin{aligned} J_\pm(1,2) \mid j_1m_1, j_2m_2\rangle = (q^{m_1}\langle j_2m_2 \pm 1 \mid J_\pm \mid j_2m_2\rangle) \mid j_1m_1, j_2m_2 \pm 1\rangle \\ + (p^{-m_2}\langle j_1m_1 \pm 1 \mid J_\pm \mid j_1m_1\rangle) \mid j_1m_1 \pm 1, j_2m_2\rangle \, . \end{aligned} \tag{4.5}$$

To calculate any power of the operators $J_\pm(1,2)$ the analogue of the binomial expansion formula in the case of the $\mathrm{SU}_{p,q}(2)$ algebra is useful. This formula

$$\begin{aligned} (J_\pm(1,2))^l &= \left(q^{J_0(1)}J_\pm(2) + J_\pm(1)p^{-J_0(2)}\right)^l \\ &= \sum_{r=0}^{l} \frac{[[l]]!}{[[l-r]]!\,[[r]]!} J_\pm^r(1)J_\pm^{l-r}(2)q^{(l-r)J_0(1)}p^{-rJ_0(2)} \end{aligned} \tag{4.6}$$

can be proved by induction. Using (4.4) and (2.12) the Casimir operator in the product space $C_2(1,2) \boxminus \Delta(C_2)$ becomes

$$\Delta(C_2) = J_-(1,2)J_+(1,2)(pq^{-1})^{J_0(1,2)} + (pq^{-1})^{J_0(1,2)-1}[[J_0(1,2)+\tfrac{1}{2}]]^2 \, . \tag{4.7}$$

The generalization of the 'vector coupling' procedure for the $\mathrm{SU}_{p,q}(2)$ algebra consists in the construction, using the tensor product basis vectors $|j_1m_1, j_2m_2\rangle$, of such linear combinations

$$|j_1j_2, jm\rangle = \sum_{m_1m_2} \langle j_1m_1, j_2m_2 \mid j_1j_2, jm\rangle \mid j_1m_1, j_2m_2\rangle \tag{4.8}$$

that belong to the IRs D^j of $\mathrm{SU}_{p,q}(2)$, i.e. that are eigenvectors of the Casimir operator (4.7) with eigenvalues $\Lambda = (pq^{-1})^{j-1}[[j+\tfrac{1}{2}]]^2$. The coefficients $\langle j_1m_1, j_2m_2 \mid j_1j_2, jm\rangle$ are the Clebsch–Gordan coefficients (CGC) for the $\mathrm{SU}_{p,q}(2)$ quantum algebra.

The standard way to calculate the CGC is to multiply both sides of the eigenvalue equation

$$C_2(1,2) \mid j_1j_2, jm\rangle = \Lambda \mid j_1j_2, jm\rangle$$

by the vector $\langle j_1m_1, j_2m_2|$, thus obtaining the recurrent relation for the CGCs

$$\begin{aligned} &\sqrt{[[j_1-m_1]]\,[[j_1+m_1+1]]\,[[j_2+m_2]]\,[[j_2-m_2+1]]} \\ &\quad \times (pq^{-1})^{\frac{1}{2}(j_1+j_2+m-1)}p^{-m_2-1}q^{m_1}\langle j_1m_1+1, j_2m_2-1 \mid j_1j_2, jm\rangle \\ &\quad + \sqrt{[[j_1+m_1]]\,[[j_1-m_1+1]]\,[[j_2-m_2]]\,[[j_2+m_2+1]]} \\ &\quad \times (pq^{-1})^{\frac{1}{2}(j_1+j_2+m-1)}p^{-m_2}q^{m_1+1}\langle j_1m_1-1, j_2m_2+1 \mid j_1j_2, jm\rangle \\ &\quad + \{\, (pq^{-1})^{(j_1-1)}p^{-m_2}q^{-m_2}[[j_1-m_1]]\,[[j_1+m_1+1]] \\ &\quad + (pq^{-1})^{(j_2-1)}p^{m_1}q^{m_1}[[j_2-m_2]]\,[[j_2+m_2+1]] \\ &\quad + (pq^{-1})^{(m-1)}[[m+\tfrac{1}{2}]]^2 - \Lambda \,\}\, \langle j_1m_1, j_2m_2 \mid j_1j_2, jm\rangle = 0 \, . \end{aligned} \tag{4.9}$$

From these relations the analytic expression for the CGCs can be found in a similar fashion, as done in the famous Racah paper [34]. Here, we prefer to apply the projection operator method for this purpose. Simultaneously it will be shown that the similar expansion (4.1) is also valid for the quantum $SU_{p,q}(2)$ algebra.

However, before turning to this point, it is pertinent to list the orthonormality relations of the CGCs

$$\sum_{m_1 m_2} \langle j_1 m_1, j_2 m_2 \mid j_1 j_2, jm\rangle \langle j_1 m_1, j_2 m_2 \mid j_1 j_2, j'm'\rangle = \delta_{j,j'}\delta_{m,m'}$$

$$|j_1 m_1, j_2 m_2\rangle = \sum_{j,m} \langle j_1 j_2, jm \mid j_1 m_1, j_2 m_2\rangle \mid j_1 j_2, jm\rangle \tag{4.10}$$

$$\sum_{j,m} \langle j_1 m_1, j_2 m_2 \mid j_1 j_2, jm\rangle \langle j_1 m_1', j_2 m_2' \mid j_1 j_2, jm\rangle = \delta_{m_1,m_1'}\delta_{m_2,m_2'}\,.$$

5. $SU_{p,q}(2)$ Clebsch–Gordan coefficients

Using the projection operator $\Delta(P^j_{m,m'}) \equiv P^j_{m,m'}(1,2)$, the vector $|j_1 j_2, jm\rangle$ can be written in the form

$$|j_1 j_2, jm\rangle = Q^{-1} P^j_{mm'}(1,2) \mid j_1 m_1', j_2 m_2'\rangle \tag{5.1}$$

where $m' = m_1' + m_2'$ and Q is a normalizing factor. Thus the CGC can be reduced to the matrix element of the projection operator:

$$\langle j_1 m_1, j_2 m_2 \mid j_1 j_2, jm\rangle = Q^{-1} \langle j_1 m_1, j_2 m_2 \mid P^j_{mm'}(1,2) \mid j_1 m_1', j_2 m_2'\rangle\,. \tag{5.2}$$

Since the values of m' and of either m_1' or m_2' are arbitrary, to simplify the calculations we set $m' = j$ and $m_1' = j_1$ hence $m_2' = j - j_1$. Then (5.2) takes the form

$$\langle j_1 m_1, j_2 m_2 \mid j_1 j_2, jm\rangle = Q^{-1} \langle j_1 m_1, j_2 m_2 | P^j_{mj}(1,2) \mid j_1 j_1, j_2 j - j_1\rangle\,.$$

Now, $|j_1 j_1\rangle$ is a highest weight vector, hence the generator $J_+(1,2)$ in $P^j_{jj}(1,2)$ becomes $J_+(2)q^{j_1}$ yielding

$$\begin{aligned}\langle j_1 m_1, j_2 m_2 \mid j_1 j_2, jm\rangle &= Q^{-1}(pq^{-1})^{\frac{1}{4}(j-m)(j-m-1)} \\ &\quad\times \left(\frac{[[j+m]]!}{[[2j]]![[j-m]]!}\right)^{\frac{1}{2}} \sum_{l\geqslant 0} (-1)^l \frac{[[2j+1]]!q^{lj_1}}{[[l]]![[2j+l+1]]!} \\ &\quad\times \langle j_1 m_1, j_2 m_2 \mid J_-^{j-m+l}(1,2) J_+^l(2) \mid j_1 j_1, j_2 j - j_1\rangle\,.\end{aligned} \tag{5.3}$$

Further using the binomial expansion (4.6) to express $J_-^{j-m+l}(1,2)$ as a power series in the generators $J_-(1)$ and $J_-(2)$ we obtain

$$
\begin{aligned}
\langle j_1 m_1, j_2 m_2 \mid j_1 j_2, jm\rangle
&= Q^{-1}(pq^{-1})^{\frac{1}{4}[(j-m)(j-m-1)+(j_1-m_1)(j_1-m_1-1)-4j_1(j-j_1)]} \\
&\quad \times p^{m_1(j-j_1)} q^{-m_2 j_1} [\![\{[[j+m]]![[2j_1]]!\} \\
&\quad / \{[[2j]]![[j-m]]![[j_1-m_1]]![[j_1+m_1]]!\}]\!]^{\frac{1}{2}} \\
&\quad \times \sum_{l\geqslant 0} (-1)^l \frac{[[2j+1]]![[j-m+l]]! p^{m_1 l} q^{l j_1} (pq^{-1})^{-j_1 l}}{[[l]]![[2j+l+1]]![[j-m+l-j_1+m_1]]!} \\
&\quad \times \langle j_2 m_2 \mid J_-^{j-m+l-j_1+m_1}(2) J_+^l(2) \mid j_2 j - j_1\rangle .
\end{aligned}
\tag{5.4}
$$

These final matrix elements can be calculated using (2.11). For the final expression we obtain

$$
\begin{aligned}
\langle j_1 m_1, j_2 m_2 \mid P^j_{mj} \mid j_1 j_1, j_2 j - j_1\rangle &= \delta_{m, m_1+m_2} p^{m_1(j-j_1)} q^{-m_2 j_1} \\
&\quad \times (pq^{-1})^{\frac{1}{2}[(m+1-j_2)(m_2-j)-j_1(m_1+j+j_2-2j_1)+m_1(m_1+1)]} \\
&\quad \times [\![\{[[2j+1]][[j+m]]![[2j_1]]![[2j+1]]![[j_1+j_2-j]]![[j_2-m_2]]!\} \\
&\quad / \{[[j_1+m_1]]![[j-m]]![[j_1-m_1]]![[j+j_2-j_1]]![[j_2+m_2]]!\}]\!]^{\frac{1}{2}} \\
&\quad \times \sum_{l\geqslant 0} (-1)^l \{[[l+j-m]]![[j+j_2-j_1+l]]! \\
&\quad \times (pq^{-1})^{\frac{1}{2}[2l(j_2-j)-l(l+1)]} p^{l m_1} q^{l j_1}\} \\
&\quad / \{[[l]]![[l+j-m_2-j_1]]![[j_1+j_2-j-l]]![[2j+l+1]]!\} .
\end{aligned}
\tag{5.5}
$$

The normalizing constant Q is calculated from

$$
Q^2 = \langle j_1 j_1, j_2 j - j_1 \mid P^j_{jm}(1,2)\, P^j_{mj}(1,2) \mid j_1 j_1, j_2 j - j_1\rangle .
$$

Since $P^j_{jm}(1,2)\, P^j_{mj}(1,2) = P^j_{jj}(1,2)$ it follows

$$
\begin{aligned}
Q^2 &= \langle j_1 j_1, j_2 j - j_1 \mid P^j_{jj}(1,2) \mid j_1 j_1, j_2 j - j_1\rangle \\
&= \langle j_1 j_1, j_2 j - j_1 \mid \sum_{l\geqslant 0} (-1)^l \frac{[[2j+1]]! q^{l j_1}}{[[l]]![[2j+l+1]]!} \\
&\quad \times J_-^l(1,2) J_+^l(2) \mid j_1 j_1, j_2 j - j_1\rangle \\
&= \langle j_2 j - j_1 \mid \sum_{l\geqslant 0} (-1)^l \frac{[[2j+1]]! q^{2l j_1}}{[[l]]![[2j+l+1]]!} \\
&\quad \times J_-^l(2) J_+^l(2) \mid j_2 j - j_1\rangle .
\end{aligned}
\tag{5.6}
$$

By adopting the standard phase convention for Q being the positive square root of Q^2 all CGCs turn out to be real. Further, it is clear that only values of the total angular momentum j satisfying the condition $|j_1 - j_2| \leqslant j \leqslant j_1 + j_2$ are allowed. We thus obtain for the quantum algebra $\mathrm{SU}_{p,q}(2)$ the standard rule for the 'vector coupling of angular momenta', i.e. expansion (4.1).

Again with (2.11) it follows for the normalizing factor,

$$Q^2 = \frac{[[j_1 + j_2 - j]]![[2j + 1]]!}{[[j_2 - j_1 + j]]!} \times \sum_{l \geqslant 0} (-1)^l \frac{[[j_2 - j_1 + j + l]]! q^{2lj_1} (pq^{-1})^{\frac{1}{2}[2l(j_1+j_2-j) - l(l+1)]}}{[[l]]![[j_1 + j_2 - j - l]]![[2j + l + 1]]!}. \tag{5.7}$$

This last sum can be calculated applying certain summation rules for some specific combinations of generalized factorials. Here we prefer the use of a recurrent approach as described in appendix B. We find

$$Q^2 = p^{(j_1+j_2-j)(j_1-j-j_2-1)} \frac{[[2j_1]]![[2j + 1]]!}{[[j_1 - j_2 + j]]![[j + j_2 + j_1 + 1]]!}. \tag{5.8}$$

With this result we find for the CGC

$$\begin{aligned}
\langle j_1 m_1, j_2 m_2 \mid j_1 j_2, jm \rangle
&= \delta_{m, m_1+m_2} p^{\frac{1}{2}[j_2(j_1+j_2+2m_1)+j_1+j_2-j(j+1)]} q^{\frac{1}{2}[j_1(j_1+j_2-2m_2)]} \\
&\quad \times (pq^{-1})^{\frac{1}{2}[m(m-j+1)-j(j_2-j_1-j)-j_1(1+j_2)-(j_2+m_1)(j_1+m_2)+j_2(j_2-1)]} \\
&\quad \times [\![\{[[2j + 1]][[j + m]]![[j_2 - m_2]]! \\
&\quad \times [[j_1 + j_2 + j + 1]]![[j_1 - j_2 + j]]![[j_1 + j_2 - j]]!\} \\
&\quad / \{[[j - m]]![[j_1 + m_1]]![[j_1 - m_1]]![[j_2 + m_2]]![[j - j_1 + j_2]]!\}]\!]^{\frac{1}{2}} \\
&\quad \times \sum_{l \geqslant 0} (-1)^{j_1+j_2-j-l} \\
&\quad \times \frac{[[2j_2 - l]]![[j_1 + j_2 - m - l]]! p^{-m_1 l} q^{-j_1 l} (pq^{-1})^{\frac{1}{2}[-l(l-2j_1-1)]}}{[[l]]![[j_1 + j_2 - j - l]]![[j_2 - m_2 - l]]![[j_1 + j_2 + j - l + 1]]!}.
\end{aligned} \tag{5.9}$$

Here the summation index $l \to j_1 + j_2 - j - l$ is adopted to obtain the expression that coincides in the classical limit $p = q = 1$ with one of standard formulae for the usual CGCs. Simple analytical formulae can be found for the important particular cases. As an example the explicit expressions for the CGCs $\langle \frac{1}{2} m_1 j_2 m_2 \mid \frac{1}{2} j_2, jm \rangle$ are given in table 1.

Table 1. The Clebsch–Gordan coefficients $\langle \frac{1}{2} m_1 j_2 m_2 \mid \frac{1}{2} j_2, jm \rangle$ for the quantum algebra $\mathrm{SU}_{p,q}(2)$.

$j = j_2 + \frac{1}{2},\ m_1 = \frac{1}{2}$	$j = j_2 - \frac{1}{2},\ m_1 = -\frac{1}{2}$
$(-1)^{\frac{1}{2}-m_1} q^{-\frac{m}{2}} p^{\frac{1}{2}[j_2+\frac{1}{2}]} (pq^{-1})^{\frac{1}{2}[m^2+j_2(j_2-2m-1)-\frac{3}{4}]} \left(\frac{[[j_2+m+\frac{1}{2}]]}{[[2j_2+1]]} \right)^{\frac{1}{2}}$	
$j = j_2 + \frac{1}{2},\ m_1 = -\frac{1}{2}$	$j = j_2 - \frac{1}{2},\ m_1 = \frac{1}{2}$
$q^{-\frac{m}{2}} p^{-\frac{1}{2}[j_2+\frac{1}{2}]} (pq^{-1})^{\frac{1}{2}[m(m+1)+j_2(j_2-2m-2)+\frac{3}{4}]} \left(\frac{[[j_2-m+\frac{1}{2}]]}{[[2j_2+1]]} \right)^{\frac{1}{2}}$	

6. Conclusion

In the literature a great amount of applications of the quantum groups have been discussed. The presence of an arbitrary deforming parameter is one of the main advantages of the quantum groups since this allows for more flexibility when dealing with applications to physical models. Along these lines of reasoning it appears that introducing more deforming parameters more useful quantum groups result. However, the number of allowed deforming parameters is limited [28]. In fact, for semi-simple algebras this number is one (Drinfeld's theorem). For the algebras $SU_{p,q}(2)$ and $SU_q(2)$ we explicitly see their equivalence from

$$[[x]] = (pq^{-1})^{-\frac{1}{2}[x-1]} \left(\frac{Q^x - Q^{-x}}{Q - Q^{-1}} \right) \qquad \text{with} \quad Q = \sqrt{pq}\,. \tag{6.1}$$

In spite of Drinfeld's theorem our results show, however, that important physical quantities, like for instance the CGCs for the $SU_{p,q}(2)$ algebra, *do depend* on the two deforming parameters p and q. There is thus a definite indication that the physical applications of the $SU_{p,q}(2)$ algebra are richer than those corresponding to the one-parameter deformed algebra.

To conclude let us point out that even if in this paper we have only considered the structure of the irreducible representations of $SU_{p,q}(2)$ and the CGC problem, it is evident that the projection operator method allows several other applications like the study of the symmetry properties of the CGCs, the explicit calculation of the Racah coefficients, of the tensor operators, the $9j$ symbols etc for this algebra.

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Appendix A. Useful relations for the p, q-numbers

The following identities can be proven by direct verification

$$[[-a]] = -(pq^{-1})^a [[a]] \tag{A1}$$

$$[[a+b]] = [[a]]\, q^b + [[b]]\, p^{-a} \tag{A2}$$

$$[[a-b]]\,[[a+b]] = [[a]]^2 - [[b]]^2 (pq^{-1})^{b-a} \tag{A3}$$

$$[[a-b]] = [[a]]\, q^{-b} - [[b]]\, p^{-a+b} q^{-b}\,. \tag{A4}$$

Appendix B. The calculation of the normalizing factor for the CGCs

To calculate

$$Q^2 = \langle j_2 j - j_1 \mid \sum_{l \geqslant 0} (-1)^l \frac{[[2j+1]]! q^{2lj_1}}{[[l]]![[2j+l+1]]!} J_-^l(2) J_+^l(2) \mid j_2 j - j_1 \rangle$$
$$\equiv \langle j_2 j - j_1 \mid P^j_{(j_1)} \mid j_2 j - j_1 \rangle \tag{B1}$$

we firstly consider

$$J_+ P^j_{(j_1)} \mid j_2 j - j_1 \rangle$$

and obtain using the permutation relation (2.10) and, substituting the operator $J_0(2)$ by its eigenvalue $j - j_1$

$$\frac{[[2j_1]]}{[[2j+2]]} p^{2(j_1 - j - 1)} P^{j+\frac{1}{2}}_{(j_1 - \frac{1}{2})} J_+ \mid j_2 j - j_1 \rangle . \tag{B2}$$

Using (2.7) we write for the vector $|j_2 j - j_1\rangle$,

$$|j_2 j - j_1\rangle = \sqrt{\frac{[[j_2 + j - j_1]]!}{[[2j_2]]![[j_1 + j_2 - j]]!}} (pq^{-1})^{-\frac{1}{4}(j_1 + j_2 - j)(j_1 + j_2 - j - 1)} J_-^{j + j_1 - j} \mid j_2 j_2 \rangle .$$

Hence

$$Q^2 = \frac{[[j_2 + j - j_1]]!}{[[j_1 + j_2 - j]]![[2j_2]]!} (pq^{-1})^{-\frac{1}{2}(j_1 + j_2 - j)(j_1 + j_2 - j - 1)}$$
$$\times \langle j_2 j_2 \mid J_+^{j_1 + j_2 - j} P^j_{(j_1)} J_-^{j_1 + j_2 - j} \mid j_2 j_2 \rangle . \tag{B3}$$

With (B2) and (2.4),

$$J_+ J_-^l \mid j_2 j_2 \rangle = [[l]] J_-^{l-1} [[2j_2 - l + 1]] (pq^{-1})^{l-1} \mid j_2 j_2 \rangle .$$

It follows that

$$\langle j_2 j_2 \mid J_+^l P^j_{j_1} J_-^l \mid j_2 j_2 \rangle = p^{2(j_1 - j - 1)} (pq^{-1})^{l-1}$$
$$\times \frac{[[2j_1]]}{[[2j+2]]} [[l]] [[2j_2 - l + 1]] \langle j_2 j_2 \mid J_+^{l-1} P^{j+\frac{1}{2}}_{(j_1 - \frac{1}{2})} J_-^{l-1} \mid j_2 j_2 \rangle . \tag{B4}$$

Iterating this we obtain

$$\langle j_2 j_2 \mid J_+^l P^j_{(j_1)} J_-^l \mid j_2 j_2 \rangle = p^{l(2j_1 - 2j - l - 1)} (pq^{-1})^{\frac{1}{2}l(l-1)}$$
$$\times \frac{[[2j_1]]![[2j_2]]![[l]]![[2j+1]]!}{[[2j_1 - l]]![[2j_2 - l]]![[2j + l + 1]]!} . \tag{B5}$$

The substitution $l = j_1 + j_2 - j$ gives the result

$$Q^2 = p^{(j_1 + j_2 - j)(j_1 - j - j_2 - 1)} \frac{[[2j_1]]![[2j+1]]!}{[[j_1 - j_2 + j]]![[j + j_2 + j_1 + 1]]!} . \tag{B6}$$

Appendix C. Particular cases of the Clebsch–Gordan coefficients

Let X_{m_1} be defined as

$$X_{m_1} \equiv \langle j_1 m_1, j_2 m_2 \mid P^j_{jj}(1,2) \mid j_1 j_1, j_2 j - j_1 \rangle = \langle j_1 m_1, j_2 m_2 \mid j_1 j_2, jj \rangle \, Q \, . \tag{C1}$$

To find a recurrent relation for X_{m_1} we write

$$|j_1 m_1\rangle = \frac{(pq^{-1})^{-\frac{1}{2}(j_1 - m_1 - 1)}}{\{[[j_1 + m_1 + 1]][[j_1 - m_1]]\}^{\frac{1}{2}}} J_-(1) \mid j_1 m_1 + 1\rangle \, . \tag{C2}$$

Noting that

$$J_+(1,2) P^j_{j,j}(1,2) = 0$$

it follows that

$$\langle j_1 m_1 | = -\frac{(pq^{-1})^{-\frac{1}{2}(j_1 - m_1 - 1)}}{\{[[j_1 + m_1 + 1]][[j_1 - m_1]]\}^{\frac{1}{2}}} \langle j_1 m_1 + 1 \mid q^{J_0(1)} J_+(2) p^{J_0(2)} \, . \tag{C3}$$

Letting $J_+(2)$ act on $\langle j_2 m_2 |$ the following recursion results:

$$X_{m_1} = -(pq^{-1})^{\frac{1}{2}[(j_2 - j_1 + m_1 - m_2 + 1)]} p^{m_2} q^{m_1 + 1} \times \left(\frac{[[j_2 + m_2]][[j_2 - m_2 + 1]]}{[[j_1 + m_1 + 1]][[j_1 - m_1]]} \right)^{\frac{1}{2}} X_{m_1 + 1} \, . \tag{C4}$$

Since $X_{j_1} = Q^2$, iteration of (C4) yields

$$X_{m_1} = (-1)^{j_1 - m_1} (pq^{-1})^{\frac{1}{2}[(j_2 + m_1 + 1)(j_1 - m_1) - (j_1 + m_1)(j_1 + m_2)]} p^{jj_1} q^{j_1 - m_1 - jm_1} \times \left(\frac{[[j_1 + m_1]]![[j_2 + m_2]]![[j_1 + j_2 - j]]!}{[[2j_1]]![[j_2 - m_2]]![[j_1 - m_1]]![[j_2 + j - j_1]]!} \right)^{\frac{1}{2}} Q^2 \, . \tag{C5}$$

We finally obtain for the CGC

$$\begin{aligned} \langle j_1 m_1, j_2 m_2 \mid j_1 j_2, jj \rangle = {} & (-1)^{j_1 - m_1} p^{\frac{1}{2}[j(m_2 - m_1 + 1) + j_1(j_1 - 1) - j_2(j_2 + 1)]} q^{j_1 - m_1} \\ & \times [\![\{ (pq^{-1})^{(j_2 - j_1 - j + 1)(j_1 - m_1)} \\ & \times [[j_1 + m_1]]![[j_2 + m_2]]![[j_1 + j_2 - j]]![[2j + 1]]! \} \\ & / \{ [[j_2 - m_2]]![[j_1 - m_1]]![[j_2 + j - j_1]]! \\ & \times [[j_1 - j_2 + j]]![[j + j_2 + j_1 + 1]]! \}]\!]^{\frac{1}{2}} \, . \end{aligned} \tag{C6}$$

Let Y_m be defined as

$$Y_m \equiv \langle j_1 j_1, j_2 m_2 \mid P^j_{mj}(1,2) \mid j_1 j_1, j_2 j - j_1 \rangle = \langle j_1 j_1, j_2 m_2 \mid j_1 j_2, jm \rangle \, Q \, . \tag{C7}$$

To find a recurrent relation for Y_m note that

$$P^j_{mj}(1,2) = \left(\frac{(pq^{-1})^{(j-m-1)}}{[[j+m+1]][[j-m]]}\right)^{\frac{1}{2}} J_-(1,2)P^j_{m+1,j}(1,2)\,. \tag{C8}$$

Letting $J_+(1,2)$ act on $\langle j_1j_1, j_2m_2|$ the following recursion results

$$Y_m = q^{j_1}\left(\frac{(pq^{-1})^{(j_2+j-m-m_2-2)}[[j_2-m_2]][[j_2+m_2+1]]}{[[j+m+1]][[j-m]]}\right)^{\frac{1}{2}} Y_{m+1}\,. \tag{C9}$$

Since $Y_j = Q^2$, iteration of (C9) yields

$$\begin{aligned} Y_m &= q^{(j-m)(m-m_2)}(pq^{-1})^{\frac{1}{2}}[(j_2-m_2)(j-m)+j_1-j+m_2] \\ &\quad\times\left(\frac{[[j+m]]![[j_2-m_2]]![[j+j_2-j_1]]!}{[[2j]]![[j-m]]![[j_2+m_2]]![[j_1+j_2-j]]}\right)^{\frac{1}{2}} Q^2\,. \end{aligned} \tag{C10}$$

We finally obtain for the CGC

$$\begin{aligned} \langle j_1j_1, j_2m_2 \mid j_1j_2, jm\rangle &= q^{-mj_1}p^{\frac{1}{2}[j(j+1)+j_1(j_1-1)-j_2(j_2+1)]} \\ &\quad\times (pq^{-1})^{\frac{1}{2}[(j_1-j_2-j+m_2+1)m_2-j(j_1+1)-j_1(j-1)+j_2(j-j_1)]}([[2j+1]])^{\frac{1}{2}} \\ &\quad\times [\![\{[[j+m]]![[2j_1]]![[j_2-m_2]]![[j-j_1+j_2]]!\} \\ &\quad/\{[[j-m]]![[j_2+m_2]]![[j_1-j_2+j]]! \\ &\quad\times [[j_1+j_2-j]]![[j_1+j_2+j+1]]!\}]\!]^{\frac{1}{2}}\,. \end{aligned} \tag{C11}$$

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